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For a through coverage of the topics you may wish to pick up a copy of [Roberts76] "Discrete Mathematical Models: With Applications to Social, Biological, and Environmental Problems" by Fred S. Roberts, 1976 Prentice Hall Inc. ISBN: 0-13-214171-X

Another book of interest (though it does not cover this topic) is "Applied Combinatorics" by Fred S Roberts, 1984 Prentice Hall Inc. ISBN: 0-13-039313-4

(Apparently a second edition of this second book was released in 2003, I am not familiar with it and do not know anyone who is)

Vulnerability is covered in Section 3.3.2 pages 98-100 of Roberts76.

So this area of Mathematics called 'Graph Theory' (which is related to another field called topology (Anyone remember our discussion of the Bridges of Konigsberg problem?) If not go here: http://mathforum.org/isaac/problems/bridges1.html) aims to study the properties of these objects we collectively call graphs. Well what is it to be a graph then, at least the graphs that are studied in graph theory (since there are other things called graphs which are very different, are we confused yet?)? Well formally a Graph G is the set {V, E} where V is the set of vertices {v1, v2, …, vi} and E is the set of Edges {e1, e2, …, ej}. An edge is a line that connects two vertices.

If you do not know, or remember, what a set is, here is a good place to learn about sets: http://mathworld.wolfram.com/Set.html

So there is a special distinction that needs to be made now. While like I said there are plenty of things called graphs (like a graph of a function) that are not studied in graph theory. There is also a similar problem in graph theory. I just told you a definition of being a graph, yet I am about to tell you there is another kind of graph called a digraph (directed graph), that has a slightly different definition and a fundamentally different property. So please pardon the confusion this so isn't my fault, but I wanted to point out that it is okay if you are slightly confused with the repetition and different meanings applied to the term graph. Just pace yourself and well if you still aren't getting it don't be afraid to ask!

So a digraph D is a graph (in the sense it too is an object we study in graph theory) and is the set {V, A} where V is still the set of vertices as previously defined, but A is the set of Arcs {a1, a2, …, ak}.

So what is the difference between an arc and an edge? Well an edge you can go freely from the vertex on one end to the other. An arc on the other hand will let you go one way from a vertex to the other, but not the other way back. Thus a directed graph's arcs point you in the direction that you can go between two vertices. So a common application of graphs and perhaps a helpful example for your imagination is a roadway, a graph G might be considered a two way street, while a digraph D could be considered a one-way street.

We’ll carry over the assumption from Roberts76 p29 that states “Unless otherwise specified, all digraphs and graphs referred to in this book have finite vertex sets. All digraphs and graphs referred to in chapters 2 and 3 have no loops. Digraphs and graphs are not allowed to have multiple arcs or edges.”.

So we need some definitions for reaching and joining properties before we can discuss vulnerability.

P 32 Table 2.1 – Reaching and Joining

	Digraph D u1, a1, … at, ut+1		Graph G u1, e1, … et, ut+1	
	Reaching	Joining		
	Path ai is (ui, ui+1)	Semipath ai is (ui, ui+1) or (ui+1, ui)	Chain ei is {ui, ui+1}	
	Simple Path Path and ui distinct	Simple Semipath Semipath and ui distinct	Simple Chain Chain and ui distinct	
	Closed Path ui+1 = ui	Close semipath ui + 1 = ui	Closed Chain ui + 1 = ui	
	Cycle (simple closed path) Path ui+1 = ui ui distinct, i <= t (ai distinct)	Semicycle (simple closed semipath) Semipath ui+1 = ui ui distinct i <= t ai distinct	Circuit (simple closed chain) Chain ui+1 = ui ui distinct i <= t ei distinct	

A path in D is a sequence: u1, a1, u2, a2, …, ut, at, at+1

Where t >= 0 each ui is in V is a vertex and each ai is in A is an arc.

A path is closed if ut+1 = u in a closed path we end at the starting point. If the all the vertices are distinct then it is a cycle. Note if vertices are distinct then arcs must be also.

The length of a path is the number of arcs in it.

We say a distance between two vertices v and u is the shortest path between them.

We say v is reachable from u if there exists a path from u to v. By definition the singleton u defines a path, so each vertex u is reachable from itself.

Theorem 2.1 – If v is reachable from u then there is a simple path from v to u.

Theorem 2.2 – If v is reachable from u and w is reachable from v then $d(u, w) \leq d(u, v) + d(v, w)$

Structural concepts – connectedness categories for digraphs

Strongly connected – if and only if for every pair of vertices u and v, v is reachable from u and u is reachable from v.

Unilaterally connected – if and only if for every pair of vertices u and v either v is reachable from u or u is reachable from v, but not necessarily both.

Weakly connected – if and only if every pair of vertices u and v is joined.

Disconnected – if and only if not weakly connected

As should be clear any higher order of connectedness has the properties of the lower one(s) except in the case of disconnected. These are typically indicated by connectedness categorical representations 0, 1, 2, 3 where 0 is disconnected and 3 is strongly connected.

For a digraph of n vertices there are $(n^2 – n)/2$ pairs of vertices.

Criteria for membership in a connectedness category

Theorem 2.3 - A digraph D is strongly connected if and only if it has a complete closed path.

Theorem 2.4 – A digraph D is unilaterally connected if and only if it has a complete path

Lemma – In any set of vertices of a unilateral digraph D there is a vertex which can reach (using arcs od D) all others in the set.

Theorem 2.5 – A digraph is weakly connected if and only if it has a complete semipath.

Note: If there are no complete semipaths then the digraph is disconnected

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So okay we now have enough definitions and hopefully understanding of the basics of graph theory to discuss vulnerability.

So let's say we have a bridge in a road system that would be considered a vulnerability point in the transportation system. Cut out the bridge and you now have two disjoint graphs when you once had a single graph with a bridge. There is also a very similar analog for any other form of network (like utility or telecommunications).

Suppose a given digraph D represents such a network and D is strongly connected. We'll say D is arc-vulnerable if and only if the removal of some arc (but not the vertices) results in a digraph which is no longer strongly connected.

So a generalization of that would be: If D is in connectedness category i, we shall say that (u, v) is an i,j-arc if removal of (u, v) results in a digraph with connectedness category j. If D has an (i, j)-arc for some j < I, we say D is arc-vulnerable.

Just like there are degrees of connectedness there are degrees of vulnerability. We generally talk about connectedness being 0, 1, 2, or 3. This leads of the following theorem and corollary:

Theorem 3.6 – If D is a strongly connected digraph, then the arc-vulnerability of D is less then or equal to the minimum indegree of a vertex of D.

Corollary – The arc-vulnerability of a strongly connected digraph is at most a/n where a is the number of arcs and n is the number of vertices.

Definition – an Indegree is the number of arcs that lead into a given vertex. For a digraph D, the digraphs indegree is the largest of these for the individual vertices.